

Gamma Distribution

Let 'X' be a +ve continuous random variable with interval $(0, \infty)$ is said to be a gamma distribution having its p.d.f:

$$f(x) = \frac{1}{\Gamma(ab^a)} x^{a-1} e^{-x/b} \quad (0 \leq x \leq \infty)$$

And gamma function is

$$\Gamma(ab^a) = \int_0^{\infty} x^{a-1} e^{-x/b} dx \quad 0 \leq x \leq \infty$$

Known as gamma function with parameter a & b.

Where a=notation/location parameter

b=scale parameter

If a=1 then it becomes exponential distribution:

$$f(x) = \frac{1}{b} e^{-x/b} \quad 0 \leq x \leq \infty$$

If b=1 then it becomes gamma distribution of single parameter:

$$f(x) = \frac{1}{\Gamma(a)} x^{a-1} e^{-x} \quad 0 \leq x \leq \infty$$

& its function is:

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx \quad 0 \leq x \leq \infty$$

If a=b=1 then it becomes standard exponential distribution:

$$f(x) = e^{-x} \quad 0 \leq x \leq \infty$$

Properties of gamma distribution

1-Area under the curve is unity.

2-Gamma distribution is a +ve continuous distribution.

3-It has two parameters a & b.

4-The range of the distribution is 0 to ∞ .

5-The mean of gamma distribution with single parameter is 'a' & its variance is also 'a'.

6-The mean of the gamma distribution with two parameters is: $E(x) = ab$ & the variance of

the gamma distribution with two parameters is: $Var(x) = ab^2$.

7-The mode of the gamma distribution is: $Mode = b(a-1)$

8- The harmonic mean of the gamma distribution is: $H.M = b(a-1)$.

9-The m.g.f of gamma distribution is: $M_x(t) = (1-bt)^{-a}$

Prove that total area under the curve is unity.

Proof:

Let by definition

$$Area = \int f(x) dx$$

As $x \sim \text{Gamma}(a, b)$

$$f(x) = \frac{1}{\Gamma(ab^a)} x^{a-1} e^{-x/b}$$

Then

$$Area = \int_0^{\infty} \frac{1}{\Gamma(ab^a)} x^{a-1} e^{-x/b} dx$$

$$Area = \frac{1}{\int_0^{\infty} ab^a x^{a-1} e^{-x/b} dx} \quad (A)$$

As we know that gamma function is:

$$\int_0^{\infty} ab^a x^{a-1} e^{-x/b} dx \quad (B)$$

Comparing (A) & (B):

$$a = a \quad \& \quad b = b$$

$$\int_0^{\infty} ab^a = \int_0^{\infty} ab^a$$

Put in (A)

$$Area = \frac{1}{\int_0^{\infty} ab^a} \int_0^{\infty} ab^a = 1 \quad \text{Hence Proved}$$

Find r^{th} moments about origin. By use it find mean & variance.

Solution:

Let by definition

$$\begin{aligned} \mu_r' &= E(x^r) \\ &= \int_0^{\infty} x^r f(x) dx = \frac{1}{\int_0^{\infty} ab^a} \int_0^{\infty} x^r x^{a-1} e^{-x/b} dx \\ &= \frac{1}{\int_0^{\infty} ab^a} \int_0^{\infty} x^{r+a-1} e^{-x/b} dx \end{aligned} \quad (A)$$

As we know that gamma function is:

$$\int_0^{\infty} ab^a x^{a-1} e^{-x/b} dx \quad (B)$$

Comparing (A) & (B)

$$a = a+r \quad \& \quad b = b$$

$$\int_0^{\infty} ab^a = \int_0^{\infty} a+rb^{a+r} \quad \text{Put in (A)}$$

$$\begin{aligned} \mu_r' &= \frac{1}{\int_0^{\infty} a.b^a} \int_0^{\infty} a+rb^{a+r} \\ \mu_r' &= \frac{1}{\int_0^{\infty} a.b^a} \int_0^{\infty} a+rb^a b^r = \frac{1}{\int_0^{\infty} a} \int_0^{\infty} a+rb^r \\ \mu_r' &= \frac{b^r \int_0^{\infty} a+rb^r}{\int_0^{\infty} a} \end{aligned} \quad (C)$$

Use r^{th} moments to find mean & variance:

$$\text{Mean} = \mu_1' = E(x)$$

Put $r = 1$ in eq (C)

$$\mu_1' = \frac{b^1 \int_0^{\infty} a+1}{\int_0^{\infty} a} = \frac{ba \int_0^{\infty} a}{\int_0^{\infty} a} = ab$$

Now, put $r = 2$ in eq.(A)

$$\mu_2 = \frac{b^2(a+2)}{a}$$

$$\mu_2 = \frac{b^2(a+1)+1}{a} = \frac{b^2(a+1)+1}{a} = \frac{b^2(a+1)a}{a} = b^2(a+1)a$$

$$\mu_2 = ab^2(a+1)$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = ab^2(a+1) - (ab)^2 = a^2b^2 + ab^2 - a^2b^2 = ab^2$$

Question

Derive m.g.f of gamma distribution. Also find mean & variance by using m.g.f.

Solution:

Let by definition

$$M_x(t) = \mu_x(t) = E(e^{tx}) = \int e^{tx} f(x) dx$$

We know that

As $x \sim \text{Gamma}(a, b)$

$$f(x) = \frac{1}{ab^a} x^{a-1} e^{-x/b}$$

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty e^{tx} x^{a-1} e^{-x/b} dx$$

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty e^{tx-x/b} x^{a-1} dx$$

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty e^{-x\left(\frac{1}{b}-t\right)} x^{a-1} dx$$

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty x^{a-1} e^{-x\left(\frac{1-bt}{b}\right)} dx$$

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty x^{a-1} e^{-\frac{x}{b}(1-bt)} dx$$

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty x^{a-1} e^{-\frac{x}{b(1-bt)}} dx \quad (A)$$

As we know that gamma function is:

$$\int_0^\infty x^{a-1} e^{-x/b} dx \quad (B)$$

Comparing (A) & (B) and we get

$$a = a \quad \& \quad b = b(1-bt)^{-1}$$

$$\int_0^\infty x^{a-1} e^{-x/b(1-bt)} dx$$

Put in (A)

$$M_x(t) = \frac{1}{ab^a} \int_0^\infty x^{a-1} e^{-x/b(1-bt)} dx$$

$$M_x(t) = \frac{1}{b^a} \left\{ b \right\}^a \left\{ (1-bt)^{-1} \right\}^a$$

$$M_x(t) = (1-bt)^{-a}$$

Required m.g.f.

Use it to find mean & variance.

$$E(x) = \mu_1' = \left[\frac{d}{dx} M_x(t) \right]_{t=0} = \left[\frac{d}{dx} (1-bt)^{-a} \right]_{t=0} = \left[-a(1-bt)^{-a-1}(-b) \right]_{t=0} = ab$$

$$E(x^2) = \mu_2' = \left[\frac{d^2}{dt^2} M_x(t) \right]_{t=0} = \frac{d}{dt} \left[\frac{d}{dt} M_x(t) \right]_{t=0}$$

$$E(x^2) = \frac{d}{dt} \left[ab(1-bt)^{-a-1} \right]_{t=0}$$

$$E(x^2) = \left[ab(-a-1)(1-bt)^{-a-2}(-b) \right]_{t=0}$$

$$E(x^2) = ab^2(a+1)$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = ab^2(a+1) - (ab)^2 = a^2b^2 + ab^2 - a^2b^2 = ab^2$$

Question

Derive Characteristic function of gamma distribution

Solution:

Let by definition

$$\theta_x(t) = M_x(it) = (1-ibt)^{-a} \quad \text{Hence the required result}$$

Question

Derive Cumulant generation function of gamma distribution

Solution:

Let by definition

$$K(t) = \log M_x(t)$$

$$K(t) = \log(1-bt)^{-a} = -a \log(1-bt)$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$K(t) = -a \left[-bt - \frac{(bt)^2}{2} - \frac{(bt)^3}{3} - \frac{(bt)^4}{4} - \dots \right]$$

$$K(t) = \left[a\beta t + a\beta^2 \frac{t^2}{2} + a\beta^3 \frac{t^3}{3} + a\beta^4 \frac{t^4}{4} + \dots \right]$$

$$K(t) = a\beta t + a\beta^2 \frac{t^2}{2!} + 2a\beta^3 \frac{t^3}{3!} + 6a\beta^4 \frac{t^4}{4!} + \dots \quad (\text{A})$$

General expression of rth cumulant

$$K(t) = K_1 t + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + K_4 \frac{t^4}{4!} + \dots \quad (\text{B})$$

Comparing (A) And (B) we get

$$K_1 = \mu_1' = \text{Mean} = a\beta$$

$$K_2 = \mu_2 = a\beta^2$$

$$K_3 = \mu_3 = 2a\beta^3$$

$$K_4 = \mu_4' = 6a\beta^4$$

$$\mu_4 = K_4 + 3K_2^2$$

$$\mu_4 = 6a\beta^4 + 3(a\beta^2)^2 = 6a\beta^4 + 3a^2\beta^4$$

$$\mu_4 = 3(2+a)a\beta^4$$

Question

Find mode of gamma distribution.

Solution:

If the two conditions are satisfied then mode exists.

$$f'(x) = 0 \quad \text{or} \quad \frac{d}{dx} \log f(x) = 0$$

$$f''(x) < 0 \quad \text{or} \quad \frac{d^2}{dx^2} \log f(x) < 0$$

As $x \sim \text{Gamma}(a, b)$

$$f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}$$

Taking log on both sides

$$\log f(x) = \log \left[\frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} \right]$$

$$\log f(x) = \left[\log \left(\frac{1}{\Gamma(a)b^a} \right) + (a-1) \log x - \frac{x}{b} \log e \right]$$

$\therefore \log_e = 1$

$$\log f(x) = \left[\log \left(\frac{1}{\Gamma(a)b^a} \right) + (a-1) \log x - \frac{x}{b} \right]$$

Differentiate w.r.t to 'x':

$$\frac{d \log f(x)}{dx} = \frac{d}{dx} \left[\log \left(\frac{1}{\Gamma(a)b^a} \right) + (a-1) \log x - \frac{x}{b} \right]$$

$$\frac{d \log f(x)}{dx} = 0 + \frac{(a-1)}{x} - \frac{1}{b}$$

$$\frac{d \log f(x)}{dx} = (a-1)x^{-1} - \frac{1}{b}$$

-----**(i)**

$$\text{Put } \frac{d \log f(x)}{dx} = 0$$

$$0 = 0 + \frac{(a-1)}{x} - \frac{1}{b}$$

$$\frac{1}{b} = \frac{(a-1)}{x}$$

$$x = b(a-1)$$

Again diff. eq (i) w.r.t to "x"

$$\frac{d^2 \log f(x)}{dx^2} = -1(a-1)x^{-2}$$

$$\frac{d^2 \log f(x)}{dx^2} = \frac{-(a-1)}{x^2}$$

$$\frac{d^2 \log f(x)}{dx^2} = \frac{-(a-1)}{b^2(a-1)^2}$$

$$\frac{d^2 \log f(x)}{dx^2} = \frac{-1}{b^2(a-1)}$$

$$\frac{d^2 \log f(x)}{dx^2} = -ve < 0 \text{ at } x = b(a-1)$$

Both conditions are satisfied then mode is existing.

$$\text{Mode} = \hat{x} = b(a-1)$$

Question

Find Harmonic mean of gamma distribution

Solution:

Let by definition

$$H.M = \frac{1}{E\left(\frac{1}{x}\right)} \quad (\text{A})$$

$$E\left(\frac{1}{x}\right) = \int \frac{1}{x} f(x) dx = \frac{1}{\overline{ab^a}} \int_0^{\infty} \frac{1}{x} x^{a-1} e^{-x/b} dx$$

$$= \frac{1}{\overline{ab^a}} \int_0^{\infty} x^{(a-1)-1} e^{-x/b} dx \quad (\text{B})$$

As we know that gamma function is

$$\overline{ab^a} = \int_0^{\infty} x^{a-1} e^{-x/b} dx \quad (\text{C})$$

Comparing (B) & (C):

a = a-1 & b = b

$$\overline{ab^a} = \overline{a-1} b^{a-1} \quad \text{Put in (B)}$$

$$E\left(\frac{1}{x}\right) = \frac{1}{\overline{ab^a}} \overline{a-1} b^{a-1} = \frac{\overline{a-1} b^a b^{-1}}{\overline{(a-1)+1} b^a} = \frac{\overline{a-1} b^{-1}}{(a-1)\overline{(a-1)}} = \frac{b^{-1}}{(a-1)} = \frac{1}{b(a-1)}$$

Put in eq (A)

$$H.M = \frac{1}{E\left(\frac{1}{x}\right)} = \frac{1}{\left(\frac{1}{b(a-1)}\right)} = b(a-1)$$

Required Result.

Question

$$\text{Prove that } \beta(a, b) = \frac{\overline{a} \overline{b}}{\overline{a+b}}$$

Solution:

We know that

$$\overline{a} = \int_0^{\infty} x^{a-1} e^{-x} dx$$

$$\overline{b} = \int_0^{\infty} y^{b-1} e^{-y} dy$$

$$\overline{a} \overline{b} = \int_0^{\infty} \int_0^{\infty} y^{b-1} x^{a-1} e^{-(x+y)} dx dy \quad (\text{A})$$

Put $z = \frac{x}{x+y}$, $s = x+y$

$z = \frac{x}{s}$, $s-x = y$

$sz = x$, $s(1-z) = y$

Partially differentiate old variables w.r.t to new variables

$$\frac{\partial y}{\partial s} = (1-z) \quad \frac{\partial x}{\partial z} = s$$

$$\frac{\partial y}{\partial z} = -s \quad \frac{\partial x}{\partial s} = z$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} z & 1-z \\ s & -s \end{vmatrix} = -zs - s(1-z) = -zs - s + zs = -s$$

$$dx.dy = |J|dsdz = sdsdz$$

Limits

$x \& y \rightarrow 0$ then $s \rightarrow 0$

$x \& y \rightarrow \infty$ then $s \rightarrow \infty$

$x \& y \rightarrow 0$ then $z \rightarrow 0$

$x \& y \rightarrow \infty$ then $z \rightarrow 1$

Put in (i)

$$\overline{a} \overline{b} = \int_0^\infty \int_0^1 (sz)^{a-1} e^{-s} (s(1-z))^{b-1} s dz ds$$

$$\overline{a} \overline{b} = \int_0^\infty \int_0^1 s^{a-1} (z)^{a-1} s^{b-1} (1-z)^{b-1} e^{-s} s dz ds$$

$$\overline{a} \overline{b} = \int_0^\infty \int_0^1 s^{a+b-2+1} (z)^{a-1} (1-z)^{b-1} e^{-s} dz ds$$

$$\overline{a} \overline{b} = \int_0^\infty s^{a+b-1} e^{-s} ds \int_0^1 (z)^{a-1} (1-z)^{b-1} dz \quad \text{(B)}$$

In right hand side 1st function is gamma function & 2nd function is beta function of kind 1st.

As we know that gamma function is:

$$\overline{a} \overline{b}^a = \int_0^\infty x^{a-1} e^{-x/b} dx \quad \text{(i)}$$

As we know that beta function is:

$$\beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx \quad \text{(ii)}$$

Comparing (B) with (i) & (ii) :

$$\overline{a} \overline{b} = \overline{a+b} \cdot \beta(a, b)$$

$$\frac{\overline{a} \overline{b}}{\overline{a+b}} = \beta(a, b) \quad \beta(a, b) = \frac{\overline{a} \overline{b}}{\overline{a+b}}$$

Hence proved.

Question

State and prove reproductive property of gamma distribution

Solution:

Statement: If $X_i (i = 1, 2, 3, \dots, n)$ are “n” independent random variables from the gamma distribution with parameter (α, β) respectively. Then show that $\sum_{i=1}^n X_i$ also follow gamma

distribution with parameter $(\sum_{i=1}^n \alpha, \beta) = (n\alpha, \beta)$

Proof: It is given that

$X_i (i = 1, 2, 3, \dots, n) \rightarrow \text{Gamma}(\alpha_i, \beta)$ and moment generation function

$$M_{x_i}(t) = (1 - \beta t)^{-\alpha_i} \quad i = 1, 2, 3, \dots, n$$

$$\text{Let } Z = \sum_{i=1}^n X_i$$

Let by definition of m.g.f

$$M_Z(t) = E(e^{tZ})$$

$$M_Z(t) = E(e^{t \sum_{i=1}^n X_i}) = E(e^{tX_1 + tX_2 + tX_3 + \dots + tX_n}) = E(e^{tX_1} e^{tX_2} e^{tX_3} \dots e^{tX_n})$$

As X's are independent then we get

$$M_Z(t) = E(e^{tX_1}) E(e^{tX_2}) E(e^{tX_3}) \dots E(e^{tX_n})$$

$$M_Z(t) = (1 - \beta t)^{-\alpha_1} (1 - \beta t)^{-\alpha_2} (1 - \beta t)^{-\alpha_3} \dots (1 - \beta t)^{-\alpha_n}$$

$$M_Z(t) = (1 - \beta t)^{-\alpha_1 - \alpha_2 - \alpha_3 - \dots - \alpha_n}$$

$$M_Z(t) = (1 - \beta t)^{-\sum_{i=1}^n \alpha_i}$$

$$M_Z(t) = (1 - \beta t)^{-n\alpha}$$

$$\text{Hence proved } \sum_{i=1}^n X_i \rightarrow \text{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$$

Question

If $X_1 \rightarrow \text{Gamma}(\alpha_1, 1)$ and $X_2 \rightarrow \text{Gamma}(\alpha_2, 1)$ then show that $X_1 + X_2$ also follows gamma distribution with parameters $(\alpha_1 + \alpha_2)$

Proof:

It is given that

$$X_1 \rightarrow \text{Gamma}(\alpha_1, 1) \text{ with m.g.f } M_{x_1}(t) = (1 - t)^{-\alpha_1}$$

Similarly

$$X_2 \rightarrow \text{Gamma}(\alpha_2, 1) \text{ with m.g.f } M_{x_2}(t) = (1 - t)^{-\alpha_2}$$

Let

$$Z = X_1 + X_2$$

Let by definition of m.g.f

$$M_Z(t) = E(e^{tZ})$$

$$M_Z(t) = E(e^{t(X_1 + X_2)})$$

$$M_Z(t) = E(e^{tX_1 + tX_2})$$

$$M_Z(t) = E(e^{tX_1} e^{tX_2})$$

As X's are independent then we get

$$M_Z(t) = E(e^{tX_1}) E(e^{tX_2})$$

$$M_Z(t) = (1 - t)^{-\alpha_1} (1 - t)^{-\alpha_2} = (1 - t)^{-(\alpha_1 + \alpha_2)}$$

$$Z = X_1 + X_2 \rightarrow \text{Gamma}(\alpha_1 + \alpha_2, 1)$$

Hence proved

Gamma distribution with single parameter

$$f(x) = \frac{1}{\Gamma(a)} x^{a-1} e^{-x} \quad (0 \leq x \leq \infty)$$

And gamma function is

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx \quad 0 \leq x \leq \infty$$

Question

Find r^{th} moments about origin. By use it finds mean & variance.

Solution:

Let by definition

$$\mu_r' = E(x^r)$$

$$\mu_r' = \int x^r f(x) dx$$

$$\mu_r' = \frac{1}{\Gamma(a)} \int_0^{\infty} x^r x^{a-1} e^{-x} dx$$

$$\mu_r' = \frac{1}{\Gamma(a)} \int_0^{\infty} x^{a+r-1} e^{-x} dx \quad (\text{A})$$

As we know that gamma function is:

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx \quad (\text{B})$$

Comparing (A) & (B)

$$a = a+r$$

$$\Gamma(a) = \Gamma(a+r) \quad \text{Put in (A)}$$

$$\mu_r' = \frac{1}{\Gamma(a)} \Gamma(a+r)$$

Put $r=1$

$$\mu_1' = \frac{1}{\Gamma(a)} \Gamma(a+1)$$

$$\mu_1' = \frac{1}{\Gamma(a)} \Gamma(a)$$

$$\mu_1' = a = \text{Mean}$$

Now, put $r = 2$ in eq.(A)

$$\mu_2' = \frac{\Gamma(a+2)}{\Gamma(a)}$$

$$\mu_2' = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)}$$

$$\mu_2' = \frac{(a+1)\Gamma(a)}{\Gamma(a)}$$

$$\mu_2' = (a+1)a = a^2 + a$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = a(a+1) - (a)^2 = a^2 + a - a^2 = a$$

Similarly obtain the others results of single parameter gamma distribution

Mode

The mode of the gamma distribution is: $\text{Mode} = (a - 1)$

Harmonic mean

The harmonic mean of the gamma distribution is: $\text{H.M} = (a - 1)$.

Moment generating function

The m.g.f of gamma distribution is: $M_x(t) = (1 - t)^{-a}$

Question

Show that $\overline{\gamma_{n+1}} = n \overline{\gamma_n}$

Solution: As we know that gamma function with single parameter

$$\overline{\gamma_\alpha} = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \text{Put } \alpha = \overline{\gamma_{n+1}}$$

$$\overline{\gamma_{n+1}} = \int_0^\infty x^{n+1-1} e^{-x} dx$$

$$\overline{\gamma_{n+1}} = \int_0^\infty x^n e^{-x} dx$$

Integrating by parts

(i) (ii)

$$\overline{\gamma_{n+1}} = x^n \int_0^\infty e^{-x} dx - \int_0^\infty e^{-x} dx \frac{dx^n}{dx}$$

$$\overline{\gamma_{n+1}} = x^n \int_0^\infty e^{-x} dx - \int_0^\infty e^{-x} dx \frac{dx^n}{dx}$$

$$\overline{\gamma_{n+1}} = x^n \frac{e^{-x}}{-1} \Big|_0^\infty - \int_0^\infty \frac{e^{-x}}{-1} nx^{n-1} dx$$

$$\overline{\gamma_{n+1}} = x^n \frac{e^{-x}}{-1} \Big|_0^\infty + \int_0^\infty e^{-x} nx^{n-1} dx$$

$$\overline{\gamma_{n+1}} = 0 + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$\overline{\gamma_{n+1}} = n \int_0^\infty x^{n-1} e^{-x} dx$$

By comparing gamma function and we get

$$\overline{\gamma_{n+1}} = n \overline{\gamma_n}$$

Hence proved